

Comparison of harmonic kernels associated to a class of semilinear elliptic equations

Mahmoud Ben Fredj

Faculté des Sciences de Monastir

Avenue de l'Environnement, 500 Monastir, Tunisia

E-mail: mahmoudbenfredj@yahoo.fr

Khalifa El Mabrouk

Département de Mathématique

Ecole Supérieure des Sciences et de Technologie de Hammam Sousse

Rue Lamine El Abbassi, 4011 Hammam Sousse, Tunisia

E-mail: khalifa.elmabrouk@fsm.rnu.tn

Abstract

Let D be a smooth domain in \mathbb{R}^N , $N \geq 3$ and let f be a positive continuous function on ∂D . Under some assumptions on φ , it is shown that the problem $\Delta u = 2\varphi(u)$ in D and $u = f$ on ∂D , admits a unique solution which will be denoted by $H_D^\varphi f$. Given two functions φ and ψ , our main goal in this paper is to investigate the existence of a constant $c > 0$ such that

$$\frac{1}{c} H_D^\varphi f \leq H_D^\psi f \leq c H_D^\varphi f.$$

1 Introduction

Let D be a bounded smooth domain in \mathbb{R}^N , $N \geq 3$. We consider the following semilinear problem

$$\begin{cases} \Delta u &= 2\varphi(u) & \text{in } D, \\ u &= f & \text{on } \partial D, \end{cases} \quad (1.1)$$

where f is a positive continuous function on ∂D . Under some conditions on φ , it will be shown that problem (1.1) admits a unique solution which will be denoted by $H_D^\varphi f$. In the particular case where $\varphi \equiv 0$, (1.1) reduces

to the classical Dirichlet problem whose the unique solution will be denoted by $H_D f$.

Given two functions φ and ψ , we say that $H_D^\varphi f$ and $H_D^\psi f$ are proportional and we write $H_D^\varphi f \approx H_D^\psi f$ if there exists $c > 0$ such that for every $x \in D$,

$$\frac{1}{c} H_D^\psi f(x) \leq H_D^\varphi f(x) \leq c H_D^\psi f(x).$$

The operators H_D^φ and H_D^ψ are said to be proportional (we write $H_D^\varphi \approx H_D^\psi$) if $H_D^\varphi f$ and $H_D^\psi f$ are proportional for every positive continuous function f on ∂D .

The main goal of this paper is to study the proportionality between H_D^φ and H_D^ψ . To this end, we shall rather compare H_D^φ to H_D . Since f is positive on ∂D , it is very simple to observe that $H_D^\varphi f \leq h$ where h is the positive harmonic function $H_D f$. Hence, the key question is whether $H_D^\varphi f \geq ch$ for some positive constant c .

There are several papers dealing with the existence of solutions to semi-linear problems which are bounded below by a harmonic function (see [1, 2, 5, 10] and their references). The second author [10] studied the problem

$$\begin{cases} \Delta u + \xi(x)\Psi(u) = 0 & \text{in } D, \\ u > h & \text{in } D, \\ u - h = 0 & \text{on } \partial D, \end{cases} \quad (1.2)$$

where $h \geq 0$ is harmonic in D , $\xi \geq 0$ is locally bounded and $\Psi > 0$ is a nonincreasing continuous function on $]0, \infty[$. He proved that (1.2) admits a unique solution provided the function

$$x \mapsto \int_D G_D(x, y) \xi(y) dy$$

is continuous on D and vanishes on ∂D , where $G_D(\cdot, \cdot)$ denotes the Green function of Δ on D (see (2.2) below).

Athreya [2] considered the problem (1.1) where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is locally Hölder continuous and decays to 0 at the same rate as t^p , $0 < p < 1$. Given a function h_0 which is continuous on \overline{D} and harmonic in D , he showed that there exists $c > 1$ such that for every continuous function f on ∂D satisfying

$$f \geq ch_0 \quad \text{on } \partial D,$$

problem (1.1) has a unique solution which is bounded below by h_0 . By probabilistic techniques, Chen, Williams and Zhao investigated in [5] the same problem where $-t \leq \varphi(t) \leq t$. They proved the existence of a solution bounded below by a positive harmonic function provided the norm of f is sufficiently small.

The problem (1.1), with $\varphi(t) = t^p$, was already studied by Atar, Athreya and Chen in [1]. They showed that the proportionality of H_D^φ and H_D holds true if $p \geq 1$. In this direction we shall prove in this paper

that $H_D^\varphi \approx H_D$ for a large class of functions φ . On the other hand, again in [1], it was conjectured that H_D^φ and H_D are not proportional when φ is given by $\varphi(t) = t^p$, $0 \leq p < 1$. Here, we shall prove this conjecture. More precisely, we give a sufficient condition on φ under which the proportionality does not hold.

After recalling in the following section some basic facts on Brownian motion, we establish in Section 3 the existence of a unique solution to problem (1.1) where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous nondecreasing and satisfies $\varphi(0) = 0$.

In Section 4 we are concerned with the proportionality between H_D^φ and the harmonic kernel H_D . We prove that the proportionality holds true provided

$$\limsup_{t \rightarrow 0} \frac{\varphi(t)}{t} < \infty, \quad (1.3)$$

and does not hold if for some $\varepsilon > 0$,

$$\int_0^\varepsilon \left(\int_0^t \varphi(s) ds \right)^{-\frac{1}{2}} dt < \infty. \quad (1.4)$$

Seeing that condition (1.4) is valid for $\varphi(t) = t^p$ with $0 \leq p < 1$, the second part of our above result gives an immediate proof of the conjecture mentioned above.

The last section will be devoted to investigate problem (1.1) in the case where the function φ is nonincreasing.

2 Preliminaries

For every subset F of \mathbb{R}^N , let $\mathcal{B}(F)$ be the set of all Borel measurable functions on F and let $\mathcal{C}(F)$ be the set of all continuous real-valued functions on F . If \mathcal{G} is a set of numerical functions then \mathcal{G}^+ (respectively \mathcal{G}_b) will denote the class of all functions in \mathcal{G} which are nonnegative (respectively bounded). The uniform convergence norm will be denoted by $\|\cdot\|$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$ be the canonical Brownian motion on the Euclidean space \mathbb{R}^N , $N \geq 3$: Ω is the set of all continuous functions from $[0, \infty[$ to \mathbb{R}^N endowed with its Borel σ -algebra \mathcal{F} . For every $t \geq 0$ and $\omega \in \Omega$,

$$X_t(\omega) = \omega(t) \quad \text{and} \quad \mathcal{F}_t := \sigma(X_s; 0 \leq s \leq t).$$

Moreover, for every $x \in \mathbb{R}^N$, P^x is the probability measure on (Ω, \mathcal{F}) under which the Brownian motion starts at x (i.e., $P^x(X_0 = x) = 1$) and $E^x[\cdot]$ denotes the corresponding expectation. Let D be a bounded domain in \mathbb{R}^N and let τ_D be the first exit time from D by X , i.e.,

$$\tau_D = \inf \{t > 0; X_t \notin D\}.$$

Let us denote by (X_t^D) the Brownian motion killed upon exiting D . It is well known that the transition density is given by

$$p^D(t, x, y) = p(t, x, y) - r^D(t, x, y); \quad t > 0, \quad x, y \in D,$$

$$\begin{aligned} \text{where} \quad p(t, x, y) &= \frac{1}{(2\pi t)^{N/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) \\ \text{and} \quad r^D(t, x, y) &= E^x [p(t - \tau_D, X_{\tau_D}, y), \tau_D < t]. \end{aligned}$$

The corresponding semigroup is then defined by

$$P_t^D f(x) = E^x [f(X_t), t < \tau_D] = \int_D p^D(t, x, y) f(y) dy, \quad x \in D,$$

for every Borel measurable function f for which this integral makes sense.

Let h be a positive harmonic function in D . We define for $x, y \in D, t > 0$,

$$p_h^D(t, x, y) = p^D(t, x, y) \frac{h(y)}{h(x)}.$$

There exists a Markov process, called the h -conditioned Brownian motion, with state space D and having p_h^D as transition density (see [4, 6, 7]). The corresponding probability measures is denoted by $(P_h^x)_{x \in D}$: for every Borel subset B of D we have

$$\begin{aligned} P_h^x(X_t \in B) &= \frac{1}{h(x)} \int_B p^D(t, x, y) h(y) dy \\ &= \frac{1}{h(x)} E^x [h(X_t), X_t \in B, t < \tau_D]. \end{aligned}$$

Besides, using the monotone class theorem, it is easily seen that for every $t > 0$ and every \mathcal{F}_t -measurable random variable $Z \geq 0$,

$$E_h^x [Z, t < \tau_D] = \frac{1}{h(x)} E^x [Z h(X_t), t < \tau_D]. \quad (2.1)$$

The open bounded subset D is called regular (for Δ) if each function $f \in \mathcal{C}(\partial D)$ admits a continuous extension $H_D f$ on \overline{D} such that $H_D f$ is harmonic in D . In other words, the function $h = H_D f$ is the unique solution to the classical Dirichlet problem

$$\begin{cases} \Delta h &= 0 & \text{in } D, \\ h &= f & \text{on } \partial D. \end{cases}$$

For every $x \in D$, the harmonic measure relative to x and D , which will be denoted by $H_D(x, \cdot)$, is defined to be the positive Radon measure on ∂D given by the mapping $f \mapsto H_D f(x)$.

In the sequel, we always assume that D is regular and let $x_0 \in D$ be a fixed point. There exists a unique function $K_D : D \times \partial D \rightarrow \mathbb{R}_+$ satisfying:

For every $z \in \partial D$, $K_D(x_0, z) = 1$.

For every $x \in D$, $K_D(x, \cdot)$ is continuous in ∂D .

For every $z \in \partial D$, $K_D(\cdot, z)$ is a positive harmonic function in D .

For every $z, w \in \partial D$ such that $z \neq w$, $\lim_{x \rightarrow w} K_D(x, z) = 0$.

We extend the function $K_D(\cdot, z)$ to $\overline{D} \setminus \{z\}$ by letting $K_D(w, z) = 0$ for every $w \in \partial D \setminus \{z\}$. The function K_D is called the Martin kernel on D .

The Green function $G_D(\cdot, \cdot)$ is defined on $D \times D$ by

$$G_D(x, y) = \int_0^\infty p^D(t, x, y) dt. \quad (2.2)$$

It is well known that G_D is continuous (in the extended sense) on $D \times D$,

$$G_D(x, y) \leq G_{\mathbb{R}^N}(x, y) = \frac{\Gamma(\frac{N}{2} + 1)}{2\pi^{\frac{N}{2}} |x - y|^{N-2}},$$

and $\lim_{x \rightarrow z} G_D(x, y) = 0$ for every $z \in \partial D$ (see [11, chapter 4]). Moreover

$$K_D(x, z) = \frac{dH_D(x, \cdot)}{dH_D(x_0, \cdot)}(z) = \lim_{y \in D, y \rightarrow z} \frac{G_D(x, y)}{G_D(x_0, y)}; \quad x \in D, z \in \partial D. \quad (2.3)$$

For $h = K_D(\cdot, z)$ where $z \in \partial D$, the h -conditioned Brownian motion will be simply called the z -Brownian motion and its transition density is given by

$$p_z^D(t, x, y) = \frac{1}{K_D(x, z)} p^D(t, x, y) K_D(y, z); \quad t > 0, x, y \in D.$$

The corresponding probability measures family will be denoted by $(P_z^x)_{x \in D}$.

3 Semilinear problem

In the sequel, we assume that $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous nondecreasing function such that $\varphi(0) = 0$. The following comparison principle will be useful to prove not only uniqueness but also the existence of a solution to problem (1.1). A more general comparison principle can be found in [10].

Lemma 3.1. *Let $\Psi \in \mathcal{B}(\mathbb{R})$ be a nondecreasing function and let $u, v \in \mathcal{C}(\overline{D})$ such that*

$$\Delta u \leq \Psi(u) \quad \text{and} \quad \Delta v \geq \Psi(v) \quad \text{in } D.$$

If $u \geq v$ on ∂D , then $u \geq v$ in D .

Proof. Define $w = u - v$ and suppose that the open set

$$\Omega = \{x \in D; w(x) < 0\}$$

is not empty. Since Ψ is nondecreasing, it is obvious that $\Delta w \leq \Psi(u) - \Psi(v) \leq 0$ in Ω , which means that w is superharmonic in Ω . Furthermore, for every $z \in \partial\Omega \cap D$ we have $w(z) = 0$ (because w is continuous in D), and for every $z \in \partial\Omega \cap \partial D$ we have $\lim_{x \in \Omega, x \rightarrow z} w(x) \geq 0$ (by hypothesis). Then $w \geq 0$ in Ω by the classical minimum principle for superharmonic functions. This yields a contradiction and therefore Ω is empty. Hence $u \geq v$ in D . \square

The Green operator in D is defined, for every Borel measurable function f for which the following integral exists, by

$$G_D f(x) = \int_D G_D(x, y) f(y) dy, \quad x \in D. \quad (3.1)$$

Hence

$$G_D f(x) = E^x \left[\int_0^{\tau_D} f(X_t) dt \right] = \int_0^\infty P_t^D f(x) dt, \quad x \in D.$$

We recall that for every $f \in \mathcal{B}_b(D)$, $G_D f$ is a bounded continuous function on D satisfying $\lim_{x \rightarrow z} G_D f(x) = 0$ for every $z \in \partial D$. Moreover, it is simple to check that

$$\Delta G_D f = -2f$$

in the distributional sense (see [6, 7]).

Lemma 3.2. *For every $M > 0$, the family $\{G_D u; \|u\| \leq M\}$ is relatively compact with respect to the uniform convergence norm.*

Proof. First, we recall that $x \mapsto G_D 1(x) = E^x[\tau_D]$ is bounded on D (see, e.g., [11, page 23]) and consequently for every u such that $\|u\| \leq M$ we get

$$\|G_D u\| \leq M \sup_{x \in D} E^x[\tau_D].$$

Thus the family $\{G_D u; \|u\| \leq M\}$ is uniformly bounded. Next, we claim that the family $\{G_D(x, \cdot); x \in D\}$ is uniformly integrable. Indeed, let $\varepsilon > 0$ and $\eta_0 > 0$. There exist $c_1 > 0$ and $c_2 > 0$ such that for every Borel subset A of D ,

$$\begin{aligned} \int_A G_D(x, y) dy &\leq c_1 \int_A \frac{dy}{|x - y|^{N-2}} \\ &\leq c_1 \int_{B(x, \eta_0)} \frac{dy}{|x - y|^{N-2}} + c_1 \int_{A \setminus B(x, \eta_0)} \frac{dy}{\eta_0^{N-2}} \\ &\leq c_2 \eta_0^2 + c_2 \frac{m(A)}{\eta_0^{N-2}}. \end{aligned}$$

Here and in all the following, m denotes the Lebesgue measure in \mathbb{R}^N . Take $\eta_0 = \sqrt{\varepsilon/2c_2}$ and $\eta = \varepsilon \eta_0^{N-2}/2c_2$. Then for every Borel subset A of D such that $m(A) < \eta$ we have

$$\int_A G_D(x, y) dy \leq \varepsilon.$$

Hence, the uniform integrability of the family $\{G_D(x, \cdot); x \in D\}$ is shown. Therefore, in virtue of Vitali's convergence theorem (see, e.g; [12]), we conclude that for every $z \in D$,

$$\begin{aligned} \lim_{x \rightarrow z} \sup_{\|u\| \leq M} \left| \int_D G_D(x, y) u(y) dy - \int_D G_D(z, y) u(y) dy \right| \\ \leq M \lim_{x \rightarrow z} \int_D |G_D(x, y) - G_D(z, y)| dy = 0. \end{aligned}$$

This means that the family $\{G_D(x, \cdot); x \in D\}$ is equicontinuous which finishes the proof of the lemma. \square

Existence of solutions to semilinear Dirichlet problems of kind (1.1) was widely studied in the literature considering various hypotheses on the function φ (see, e.g., [3, 8, 9, 10]). In our setting, we get the following theorem.

Theorem 3.3. *For every $f \in C^+(\partial D)$, there exists one and only one function $u \in C^+(\overline{D})$ satisfying problem (1.1). Furthermore, a bounded Borel function u on D is a solution to (1.1) if and only if $u + G_D \varphi(u) = H_D f$.*

Proof. By a classical computation, it is not hard to establish the second part of the theorem. We also observe that, by the comparison principle (Lemma 3.1), problem (1.1) possesses at most one solution. So, it remains to prove the existence of a solution to (1.1). Take $f \in C^+(\partial D)$, $a = \|f\|$, $M = a + \varphi(a) \sup_{x \in D} E^x[\tau_D]$ and define $\Lambda = \{u \in C(\overline{D}); \|u\| \leq M\}$. Let $h = H_D f$ and consider the operator $T : \Lambda \rightarrow C(\overline{D})$ defined by

$$Tu(x) = h(x) - E^x \left[\int_0^{\tau_D} g(u(X_s)) ds \right], \quad x \in D,$$

where g is the real-valued odd function given by $g(t) = \inf(\varphi(t), \varphi(a))$ for every $t \geq 0$. Since $|g(t)| \leq \varphi(a)$ for every $t \in \mathbb{R}$, we get

$$|Tu(x)| \leq M$$

for every $x \in D$ and every $u \in \Lambda$. This implies that $T(\Lambda) \subset \Lambda$. Now, let $(u_n)_{n \geq 0}$ be a sequence in Λ converging uniformly to $u \in \Lambda$. Let $\varepsilon > 0$. Since g is uniformly continuous in $[-M, M]$, we deduce that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $s \in [0, \tau_D]$

$$|g(u_n(X_s)) - g(u(X_s))| < \varepsilon.$$

It follows that for every $n \geq n_0$ and $x \in D$,

$$\begin{aligned} |Tu_n(x) - Tu(x)| &= \left| E^x \left[\int_0^{\tau_D} g(u_n(X_s)) ds \right] - E^x \left[\int_0^{\tau_D} g(u(X_s)) ds \right] \right| \\ &\leq E^x \left[\int_0^{\tau_D} |g(u_n(X_s)) - g(u(X_s))| ds \right] \\ &\leq \varepsilon \sup_{x \in D} E^x[\tau_D]. \end{aligned}$$

This shows that $(Tu_n)_{n \geq 0}$ converges uniformly to Tu . We then conclude that T is a continuous operator. On the other hand, Λ is a closed bounded convex subset of $\mathcal{C}(\overline{D})$. Moreover, in virtue of Lemma 3.2, $T(\Lambda)$ is relatively compact. Thus, the Schauder's fixed point theorem ensures the existence of a function $u \in \Lambda$ such that $u = h - G_D g(u)$. Applying the comparison principle, we obtain that $0 \leq u \leq a$ and so $g(u) = \varphi(u)$. Hence, the proof is finished. \square

The unique solution to problem (1.1) will be always denoted by $H_D^\varphi f$. However, in the particular case where $\varphi : t \mapsto t^p, p > 0$, we may write $H_D^p f$ instead of $H_D^\varphi f$.

4 Proportionality of $H_D^\varphi f$ and $H_D f$

In the sequel, we suppose that D is a Lipschitz bounded domain of \mathbb{R}^N . We recall the Feynman-Kac theorem (see [6, Theorem 4.7]) which plays an important role in what follows: for every $f \in \mathcal{C}^+(\partial D)$ and $q \in \mathcal{B}_b^+(D)$, the function $v \in \mathcal{C}(\overline{D})$ given by

$$v(x) = E^x \left[f(X_{\tau_D}) \exp \left(- \int_0^{\tau_D} q(X_s) ds \right) \right], \quad x \in D, \quad (4.1)$$

is the unique solution of the problem

$$\begin{cases} \Delta v &= 2qv & \text{in } D, \\ v &= f & \text{on } \partial D. \end{cases}$$

Let us notice that v given by (4.1) satisfies the following integral equation:

$$v(x) = h(x) - \int_D G_D(x, y) q(y) v(y) dy, \quad x \in D.$$

Our first result in this section is the following:

Theorem 4.1. *Assume that*

$$\limsup_{t \rightarrow 0} \frac{\varphi(t)}{t} < \infty. \quad (4.2)$$

Then $H_D^\varphi f \approx H_D f$ for every function $f \in \mathcal{C}^+(\partial D)$.

Proof. Let $f \in \mathcal{C}^+(\partial D)$ be nontrivial, that is $h = H_D f > 0$ in D . Let $u = H_D^\varphi f$ and define

$$q := \frac{\varphi(u)}{u} 1_{\{u > 0\}}.$$

Then q is a positive bounded function in D by (4.2), and u satisfies the problem

$$\begin{cases} \Delta u &= 2qu & \text{in } D, \\ u &= f & \text{on } \partial D. \end{cases} \quad (4.3)$$

We define

$$w(x, z) = E_z^x \left[\exp \left(- \int_0^{\tau_D} q(X_t) dt \right) \right], \quad x \in D, z \in \partial D.$$

By Feynman-Kac theorem and [6, Proposition 5.12], we have

$$\begin{aligned} u(x) &= E^x \left[f(X_{\tau_D}) \exp \left(- \int_0^{\tau_D} q(X_s) ds \right) \right] \\ &= \int_{\partial D} w(x, z) f(z) H_D(x, dz). \end{aligned} \quad (4.4)$$

Since for every $x \in D$

$$E^x \left[\exp \left(- \int_0^{\tau_D} q(X_s) ds \right) \right] < \infty,$$

by [6, Theorem 7.6] there exists $c > 0$ such that

$$\frac{1}{c} \leq w(x, z) \leq c, \quad x \in D, z \in \partial D. \quad (4.5)$$

Combining (4.4) and (4.5) we conclude that for every $x \in D$,

$$\begin{aligned} \frac{1}{c} H_D f(x) &= \frac{1}{c} \int_{\partial D} f(z) H_D(x, dz) \\ &\leq u(x) \\ &\leq c \int_{\partial D} f(z) H_D(x, dz) = c H_D f(x). \end{aligned}$$

Hence, $H_D f \approx H_D^\varphi f$. □

Let us notice that the hypothesis mentioned in the previous theorem will be trivially satisfied provided the function $t \mapsto \varphi(t)/t$ is nondecreasing or if it is bounded and nonincreasing on $]0, \infty[$. In particular, it follows that $H_D^\varphi f \approx H_D f$ for every function $f \in \mathcal{C}^+(\partial D)$ if the function φ is given

$$\varphi(t) = t^p \quad \text{with } p \geq 1 \quad \text{or} \quad \varphi(t) = \log(1 + t).$$

We shall write $H_D^\varphi \approx H_D$ if $H_D^\varphi f \approx H_D f$ for every function $f \in \mathcal{C}^+(\partial D)$. Hence, by Theorem 4.1, $H_D^p \approx H_D$ for every $p \geq 1$. This was established by Atar, Athreya and Chen in [1]. In the same paper, the authors conjectured that $H_D^p \not\approx H_D$ for every $0 < p < 1$. In the following, we shall prove this conjecture. More precisely, we give a sufficient condition on φ under which $H_D^\varphi \not\approx H_D$.

Theorem 4.2. *Assume that there exists $\varepsilon > 0$ such that*

$$\int_0^\varepsilon \left(\int_0^s \varphi(r) dr \right)^{-\frac{1}{2}} ds < \infty. \quad (4.6)$$

Then there exists $f \in \mathcal{C}^+(\partial D)$ such that $H_D^\varphi f \not\approx H_D f$.

Proof. We easily observe, in virtue of condition (4.6), that the function Q defined for every $t \geq 0$ by

$$Q(t) = \frac{1}{2} \int_0^t \left(\int_0^s \varphi(r) dr \right)^{-\frac{1}{2}} ds$$

is increasing continuous on $[0, \infty[$, twice differentiable on $]0, \infty[$ and invertible from $[0, \infty[$ to $[0, \bar{\rho}[$ where $\bar{\rho} := \lim_{t \rightarrow \infty} Q(t)$ (notice that $\bar{\rho}$ may be infinite). Without loss of generality, we assume that

$$D \subset \mathbb{R}_+^N \cap B(0, \rho) \quad \text{and} \quad 0 \in \partial D,$$

where $\mathbb{R}_+^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N; x_1 > 0\}$ and $0 < \rho < \bar{\rho}$. Consider the function u defined by $u(x) = R(x_1)$ for every $x = (x_1, \dots, x_N) \in \bar{D}$ where R denotes the inverse function of Q . Then, it is obvious that

$$u \in \mathcal{C}(\bar{D}) \cap \mathcal{C}^2(D).$$

Moreover, an elementary calculus yields that for every $x \in D$,

$$\Delta u(x) = R''(x_1) = 2\varphi(R(x_1)) = 2\varphi(u(x)).$$

Hence $u = H_D^\varphi f$ where $f = u|_{\partial D}$. Let $h = H_D f$ and consider the harmonic function $g : x \mapsto x_1$. From the boundary Harnack principle it follows that there exists an open neighborhood V of 0 such that

$$g \approx h \quad \text{in } V \cap D.$$

On the other hand

$$\lim_{x \rightarrow 0} \frac{u(x)}{g(x)} = \lim_{t \rightarrow 0} \frac{t}{Q(t)} = 0.$$

Thus $u \not\approx g$ and consequently $u \not\approx h$. □

Since the function $\varphi : t \mapsto t^p$ satisfies (4.6) for $0 < p < 1$, we deduce from the previous theorem that, for small p , $H_D^p \not\approx H_D$ which proves the conjecture given in [1].

In the remainder of this section, we shall proceed to answer the following question: in the case where (4.2) fails, for which function $f \in \mathcal{C}^+(\partial D)$, the proportionality of $H_D f$ and $H_D^\varphi f$ does hold?

First, the following proposition is easily obtained.

Proposition 4.3. *Let $f \in \mathcal{C}^+(\partial D)$, $h = H_D f$ and $u = H_D^\varphi f$. If*

$$\sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y) \varphi(h(y)) dy < 1, \quad (4.7)$$

then $u \approx h$.

Proof. It is an immediate consequence of the formula $h = u + G_D \varphi(u)$ and the fact that φ is nondecreasing. □

Hence, one direction in solving the question above is to investigate functions f for which condition (4.7) is fulfilled. Let us notice that " < 1 " in (4.7) can not be replaced by " $< \infty$ ". In fact, as will be shown below, for smooth domain D we always have

$$\sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y) \varphi(h(y)) dy < \infty. \quad (4.8)$$

However, for $\varphi(t) = t^p$ with $0 < p < 1$, Theorem 4.2 proves that there exists a function $f \in \mathcal{C}^+(\partial D)$ such that u and h are not proportional.

From now on, we assume that D is a bounded $C^{1,1}$ domain. Let $\delta(x) := \inf_{z \in \partial D} |x - z|$ be the Euclidean distance from $x \in D$ to the boundary of D . We recall that Zhao [13] established the following:

$$G_D(x, y) \approx \min \left\{ \frac{1}{|x - y|^{N-2}}, \frac{\delta(x)\delta(y)}{|x - y|^N} \right\}. \quad (4.9)$$

Lemma 4.4. *For every positive harmonic function h in D , there exists a positive constant c such that for every $x \in D$,*

$$\int_D G_D(x, y) dy \leq c h(x).$$

Proof. Let h be a positive harmonic function in D . Since the Euclidian boundary of D coincides with the Martin one, there exists a positive Borel measure ν on ∂D such that

$$h = \int_{\partial D} K_D(\cdot, z) d\nu(z). \quad (4.10)$$

We claim that there exists $C > 0$ such that for every $x \in D$ and $z \in \partial D$,

$$h(x) \geq C \delta(x). \quad (4.11)$$

Indeed, let $x \in D$ and $z \in \partial D$. Then, it is simple to observe that $\delta(x)\delta(y) \leq |x - y|^2$ for every $y \in D$ such that $8|y - z| < \delta(x)$. Hence, by (4.9) there exists a constant $c_1 > 0$ such that for every $y \in D \cap B(z, \delta(x)/8)$,

$$G_D(x, y) \geq c_1 \frac{\delta(x)\delta(y)}{|x - y|^N}.$$

Again by (4.9) there exists $c_2 > 0$ such that

$$G_D(x_0, y) \leq c_2 \frac{\delta(x_0)\delta(y)}{|x_0 - y|^N},$$

where x_0 denotes, as was mentioned in Section 2, a reference point. Therefore for every $y \in D \cap B(z, \delta(x)/8)$,

$$\frac{G_D(x, y)}{G_D(x_0, y)} \geq \frac{c_1 |x_0 - y|^N \delta(x)\delta(y)}{c_2 \delta(x_0)\delta(y) |x - y|^N}.$$

Whence, letting y tend to z we obtain that

$$K_D(x, z) \geq c_3 \frac{\delta(x)}{|x - z|^N}$$

where c_3 is a positive constant not depending on x and z . This and formula (4.10) yield (4.11). On the other hand, in [13] it is shown that there exists $c_4 > 0$ such that for every $x, y \in D$,

$$G_D(x, y) \leq c_4 \frac{\delta(x)}{|x - y|^{N-1}}.$$

Hence, using (4.11) it follows that

$$\sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y) dy = \frac{c_4}{C} \sup_{x \in D} \int_D \frac{dy}{|x - y|^{N-1}} < \infty.$$

□

Theorem 4.5. *Assume that*

$$\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = 0. \quad (4.12)$$

Then for every $f \in C^+(\partial D)$, there exists a positive constant α_f such that $H_D^\varphi(\alpha f) \approx H_D(\alpha f)$ for every $\alpha \geq \alpha_f$.

Proof. Let $f \in C^+(\partial D)$ be non trivial and let $h = H_D f$. By the previous lemma, there exists $c > 0$ (depending on h) such that for every $\alpha > 0$ and every $x \in D$,

$$G_D \varphi(\alpha h)(x) \leq \varphi(\alpha \|h\|) G_D 1(x) \leq c \varphi(\alpha \|h\|) h(x).$$

Therefore

$$\sup_{x \in D} \frac{G_D \varphi(\alpha h)(x)}{\alpha h(x)} \leq c \frac{\varphi(\alpha \|h\|)}{\alpha}.$$

On the other hand, by (4.12) there exists $A > 0$ such that for every $t \geq A$,

$$\frac{\varphi(t)}{t} < \frac{1}{c \|h\|}.$$

Take $\alpha_f := A / \|h\|$. Then for every $\alpha \geq \alpha_f$, we have

$$\sup_{x \in D} \frac{G_D \varphi(\alpha h)(x)}{\alpha h(x)} < 1,$$

which yields, by Proposition 4.3, that $H_D^\varphi(\alpha f) \approx H_D(\alpha f)$. □

5 More about problem (1)

This last section is devoted to investigate the problem (1) in the case where φ is nonincreasing. Let us notice that, in this setting, we do not guarantee the existence nor the uniqueness of the solution to problem (1) and hence the operator H_D^φ is no longer defined. As above, we assume that D is a $C^{1,1}$ bounded domain of \mathbb{R}^N , $N \geq 3$.

Theorem 5.1. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous nonincreasing function, $f \in C^+(\partial D)$ and let $h = H_D f$ such that*

$$\sup_{x \in D} E_h^x \left[\int_0^{\tau_D} \frac{1}{h(X_s)} ds \right] \leq \frac{1}{e \varphi(0)}. \quad (5.1)$$

Then the problem (1.1) possesses a solution $u \in C^+(\overline{D})$ satisfying $u \approx h$.

Proof. Of course we assume that $\varphi(0) > 0$ and f is non trivial. By hypothesis,

$$c := \sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y) dy \leq \frac{1}{e \varphi(0)}.$$

It is easily seen that there exists $b > 0$ such that $e^b \varphi(0) c = b$. We observe that the set

$$\Lambda = \left\{ u \in C(\overline{D}); e^{-b} h \leq u \leq h \right\}$$

is closed bounded and convex in $C(\overline{D})$. Consider $T : \Lambda \rightarrow C(\overline{D})$ defined by

$$Tu(x) = E^x \left[h(X_{\tau_D}) \exp \left(- \int_0^{\tau_D} \frac{\varphi(u(X_s))}{u(X_s)} ds \right) \right], \quad x \in D.$$

Then, it is clear that $Tu \leq h$ for every $u \in \Lambda$. Furthermore, for every $x \in D$,

$$\begin{aligned} \frac{Tu(x)}{h(x)} &= E_h^x \left[\exp \left(- \int_0^{\tau_D} \frac{\varphi(u(X_s))}{u(X_s)} ds \right) \right] \\ &\geq E_h^x \left[\exp \left(- e^b \varphi(0) \int_0^{\tau_D} \frac{1}{h(X_s)} ds \right) \right] \\ &\geq \exp \left(- e^b \varphi(0) E_h^x \left[\int_0^{\tau_D} \frac{1}{h(X_s)} ds \right] \right) \\ &\geq \exp \left(- e^b \varphi(0) c \right) \\ &= \exp(-b). \end{aligned}$$

This yields that $T(\Lambda) \subset \Lambda$. On the other hand, for every $u \in \Lambda$ we have

$$e^{-b \frac{\varphi(u)}{u}} Tu \leq \varphi(0).$$

So, in virtue of Lemma 3.2, we deduce that the family

$$\left\{ \int_D G_D(\cdot, y) \frac{\varphi(u(y))}{u(y)} Tu(y) dy; u \in \Lambda \right\}$$

is relatively compact in $\mathcal{C}(\overline{D})$. Seeing that

$$Tu(x) + \int_D G_D(x, y) \frac{\varphi(u(y))}{u(y)} Tu(y) dy = h(x), \quad x \in D,$$

we conclude that $T(\Lambda)$ is relatively compact in $\mathcal{C}(\overline{D})$ and consequently T is continuous. Then by Schauder's fixed point theorem, there exists $u \in \Lambda$ such that

$$u(x) + \int_D G_D(x, y) \varphi(u(y)) dy = h(x), \quad x \in D.$$

Hence, u is a solution to the problem (1). Moreover $u \approx h$ since $u \in \Lambda$. \square

Corollary 5.2. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous nonincreasing function. For every $f \in C^+(\partial D)$, there exists $\alpha_f > 0$ such that for every $\alpha \geq \alpha_f$ the problem*

$$\begin{cases} \Delta u &= 2\varphi(u) & \text{in } D, \\ u &= \alpha f & \text{on } \partial D, \end{cases}$$

admits a solution $u \in C^+(\overline{D})$ satisfying $u \approx H_D f$.

Proof. Let $f \in C^+(\partial D)$ be non trivial and let $h = H_D f$. It suffices to consider

$$\alpha_f = e\varphi(0) \sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y) dy$$

and apply the previous theorem for αf , $\alpha \geq \alpha_f$. \square

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